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An Algorithm for Solving Boundary Value Problems

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In this paper a numerical algorithm, based on the decomposition technique, is presented for solving a class of nonlinear boundary value problems. The method is implemented for well-known examples, including Troesch's and Bratu's problems which have been extensively studied. The scheme is shown to be highly accurate, and only a few terms are required to obtain accurate computable solutions.

Key Words: Bratu's problem; Troesch's problem; decomposition method.

1. INTRODUCTION

The Adomian polynomial algorithm has been extensively used to solve linear and nonlinear problems arising in many interesting applications (see, for example, [3, 4, 11, 12, 13, 19]). The algorithm (a decomposition method) assumes a series solution for the unknown quantity. It has been shown [10] that the series converges fast, and with only a few terms this series approximates the exact solution with a fairly reasonable error, normally less than 1%. In this paper, we shall adapt the algorithm to the solution of boundary value problems arising in the modeling of interesting applications. The idea here is to obtain the integral representation of the boundary value problem through the construction of the underlying Green's function. We will adapt the decomposition method to the integral formulation

$$u(x) = \int_a^b g(x,s)F(u(s))\,ds + f(x) \tag{1.1}$$

and analyze the solution. In (1.1), g, F, and f are known functions.

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The balance of this paper is as follows. In Section 2, we give a brief description of the decomposition method. In Section 3, we will describe the general algorithm as it applies to the solution of boundary value problems. In Section 4, we adapt the algorithm for some examples of the boundary value problems. In particular, we will consider Troesch's problem [21] and Bratu's problem [8]. A brief discussion of these problems will also be given in Section 4.

2. ANALYSIS

In this section we first describe the algorithm of the decomposition method as it applies to a general nonlinear equation of the form

$$u - N(u) = f, \tag{2.1}$$

where N is a nonlinear operator on a Hilbert space H and f is a known element of H. We assume that for a given f a unique solution u of (2.1) exists.

The decomposition method assumes a series solution for u given by

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots \tag{2.2}$$

and the nonlinear operator N can be decomposed into

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where the A_n's are the Adomian polynomials of $u_0, \ldots, u_n$ given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0} \qquad n = 0, 1, \ldots . \tag{2.4}$$

Substituting equations (2.2) and (2.3) into the functional equation (2.1) yields

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f. \tag{2.5}$$

If the series in (2.5) is convergent, then (2.5) holds upon setting

$$\begin{aligned} u_0 &= f \\ u_1 &= A_0(u_0) \\ u_2 &= A_1(u_0, u_1) \\ &\vdots \\ u_n &= A_{n-1}(u_0, u_1, \ldots, u_{n-1}) \\ &\vdots \end{aligned} \tag{2.6}$$

Thus, one can recursively determine every term of the series $\sum_{n=0}^{\infty} u_n$. The convergence of this series has been established (see [7]). The two hypotheses necessary for proving convergence of the decomposition method as given in [7] are:

Condition 1. The nonlinear functional equation (2.1) has a series solution $\sum_{n=0}^{\infty} u_n$ such that $\sum_{n=0}^{\infty}(1+\epsilon)^n|u_n| < \infty$, where $\epsilon > 0$ may be very small.

Condition 2. The nonlinear operator $N(u)$ can be developed in the series $N(u) = \sum_{n=0}^{\infty} \alpha_n u^n$.

These hypotheses, for proving convergence, are generally satisfied in physical problems.

To illustrate the scheme, let the nonlinear operator $N(u)$ be a nonlinear function of u, say $g(u)$. Assume that the Taylor expansion of $g(u)$ around u_0 is

$$g(u) = g(u_0) + g^{(1)}(u_0)(u - u_0) + \frac{1}{2!}g^{(2)}(u_0)(u - u_0)^2 + \cdots. \tag{2.7}$$

Substituting the difference $u - u_0$ from Eq. (2.2) into Eq. (2.7), we get

$$g(u) = g(u_0) + g^{(1)}(u_0)(u_1 + u_2 + \cdots) + \frac{1}{2!}g^{(2)}(u_0)(u_1 + u_2 + \cdots)^2 + \cdots.$$

After expanding, this results in

$$\begin{aligned} g(u) = {} & g(u_0) + g^{(1)}(u_0)(u_1 + u_2 + \cdots) \\ & + \frac{1}{2!}g^{(2)}(u_0)\left(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + 2u_2u_3 + u_3^2 + \cdots\right) \\ & + \frac{1}{3!}g^{(3)}(u_0)\left(u_1^3 + 3u_1^2u_2 + 3u_1^2u_3 + 3u_1u_2^2 + 3u_1u_3^2 + \cdots\right) + \cdots. \end{aligned} \tag{2.8}$$

Adomian polynomials are obtained by a reordering and rearranging of the terms given in Eq. (2.8). Indeed, to determine the Adomian polynomial, one needs to determine the order of each term in Eq. (2.8), which actually depends on both the subscripts and the exponents of the u_n's. To be more specific, we define the order of the component u_l^m to be ml, and $u_l^m u_j^n$ to be $ml + nj$. Then the Adomian polynomial A_0 depends on u_0 with order 0, A_1 depends upon u_0 and u_1 with order 1, etc. Therefore, rearranging the terms in the expansion Eq. (2.8) according to the order, and assuming that $N(u)$ is as given in Eq. (2.3), will give A_n as

$$\begin{aligned} A_0 &= g(u_0) \\ A_1 &= u_1 g^{(1)}(u_0) \\ A_2 &= u_2 g^{(1)}(u_0) + \frac{u_1^2}{2!}g^{(2)}(u_0) \\ A_3 &= u_3 g^{(1)}(u_0) + u_1u_2 g^{(2)}(u_0) + \frac{u_1^3}{3!}g^{(3)}(u_0) \\ &\vdots \end{aligned} \tag{2.9}$$

Once the A_n are determined by Eq. (2.9), one can recurrently determine the terms u_n of the series from Eq. (2.6), and hence the solution u. It is easy to verify that when $N(u)$ is $g(u)$, formula (2.4) yields the same result as in (2.9).

For a detailed description of the decomposition method, we refer the reader to [2–7].

Since we will form the integral representation of boundary value problems in Section 4, we describe the application of the decomposition method to an integral equation of the form

$$u(x) = \int_a^b g(x,s)F(u(s))\,ds + f(x), \tag{2.10}$$

where $g(x,s)$ is referred to as the kernel, F is a nonlinear function of u, and $f(x)$ is a given function.

Assuming $F(u)$ is analytic (and thus satisfies Condition 2), we can write

$$F(u) = \sum_{k=0}^{\infty} A_k(u_0, u_1, \ldots, u_k), \tag{2.11}$$

where A_k are the specially generated Adomian polynomials given by (2.4). We note that the expansion $A_0,\ A_1,\ A_2, \ldots$ is valid in general when the nonlinearity, $F(u)$, admits a Taylor expansion at u_0 (see [16]).

Substituting Eqs. (2.2) and (2.11) into Eq. (2.10), we have

$$u_0 + u_1 + u_2 + \cdots = \int_a^b g(x,s)(A_0 + A_1 + A_2 + \cdots)\,ds + f(x).$$

If the series is convergent, then we can determine each term of the series $\sum_{n=0}^{\infty} u_n$ recursively:

$$\begin{aligned}
u_0 &= f(x),\\
u_1 &= \int_a^b g(x,s)A_0(u_0)\,ds,\\
u_2 &= \int_a^b g(x,s)A_1(u_0,u_1)\,ds\\
&\cdots\\
u_n &= \int_a^b g(x,s)A_{n-1}(u_0,u_1,\ldots,u_{n-1})\,ds,\\
&\cdots
\end{aligned} \tag{2.12}$$

The algorithm in (2.12) determines the u_i's and hence the solution u can determined by Eq. (2.2). The decomposition method can be applied to solve problems in higher dimensions (see [6, 7]). We will specify how Conditions 1 and 2 are satisfied for the examples that will be presented in Section 4.

3. APPLYING THE DECOMPOSITION METHOD TO BOUNDARY VALUE PROBLEMS

In this section we consider boundary value problems of the form

$$\begin{aligned}
-u'' &= \lambda F(u)\\
u(0) = \alpha, &\qquad u(1) = \beta,
\end{aligned} \tag{3.1}$$

where $\lambda > 0$ and the nonlinear function $F(u)$ is assumed to have a power series representation in accordance with Condition 2.

The Green's function of (3.1) is well known (see [20, 23]) and is given by

$$g(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1 \end{cases}. \tag{3.2}$$

Problem (3.1) can then be represented in an integral form as

$$u(x) = \lambda \int_0^1 g(x,s)F(u(s))\,ds + (1-x)\,\alpha + x\beta. \tag{3.3}$$

The nonlinear equation in (3.3) will be solved using the decomposition method as in Section 2. Again, we assume a series solution for (3.3),

$$u = \sum_{i=0}^{\infty} u_i, \tag{3.4}$$

which is convergent if a condition like Condition 1 is met. The nonlinear function $F(u)$ is

$$F(u) = \sum_{i=0}^{\infty} A_i, \tag{3.5}$$

where A_i are the Adomian polynomials constructed in the way explained in Eq. (2.9). If $F(u)$ has a Taylor expansion at u_0,

$$F(u) = F(u_0) + F'(u_0)(u-u_0) + \frac{F''(u_0)}{2!}(u-u_0)^2 + \frac{F^{(3)}(u_0)}{3!}(u-u_0)^3 + \cdots$$

with $u - u_0 = u_1 + u_2 + \cdots$, then the Adomian polynomials $A_0,\ A_1,\ A_2,\ \ldots$ are given by

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1 F'(u_0),\\
A_2 &= u_1^2 F''(u_0)\big/2! + u_2 F'(u_0),\\
A_3 &= u_3 F'(u_0) + 2u_1u_2F''(u_0)/2! + u_1^3 F^{(3)}(u_0)\big/3!,\\
A_4 &= u_4 F'(u_0) + \left(2u_1u_3 + u_2^2\right)F''(u_0)\big/2! + 3u_1^2u_2F^{(3)}(u_0)\big/3! + u_1^4F^{(4)}(u_0)\big/4!,\\
&\vdots
\end{aligned}$$

As we noted earlier, the expansions $A_0,\ A_1, \ldots$ are valid in general when $F(u)$ admits a Taylor expansion at u_0.

It follows from the series solution and Eq. (3.3) that

$$\begin{aligned}
u(x) = \sum_{i=0}^{\infty} u_i &= \lambda \int_0^1 g(x,s)\sum_{i=0}^{\infty} A_i\,ds + (1-x)\alpha + x\beta\\
&= \lambda \sum_{i=0}^{\infty} \int_0^1 g(x,s)A_i\,ds + (1-x)\alpha + x\beta.
\end{aligned} \tag{3.6}$$

Equating each term yields

$$
\begin{aligned}
u_0 &= (1-x)\alpha + x\beta, \\
u_1 &= \lambda \int_0^1 g(x,s) A_0 \, ds \\
&\vdots \\
u_{n+1} &= \lambda \int_0^1 g(x,s) A_n \, ds.
\end{aligned}
\tag{3.7}
$$

Since all the u_i's are known, the solution $u = u_0 + u_1 + u_2 + \cdots$ to Eq. (3.3) is determined.

4. EXAMPLES

In this section we apply the algorithm described in the previous section to some examples of boundary value problems.

EXAMPLE 1 (Troesch's problem). In this example we consider the boundary value problem, Troesch's problem,

$$u'' = \lambda \sinh \lambda u, \qquad 0 \le x \le 1 \tag{4.1}$$

with the boundary conditions $u(0) = 0,\ u(1) = 1$.

Troesch's problem was described and solved by Weibel [22]. It arises from a system of nonlinear ordinary differential equations which occur in an investigation of the confinement of a plasma column by radiation pressure. The problem has been studied extensively. Troesch found its numerical solution by the shooting method (see [21]). The closed form solution to this problem in terms of the Jacobian elliptic function has been given in [18] as

$$u(x) = \frac{2}{\lambda} \sinh^{-1}\left\{ \frac{\dot{u}(0)}{2} \mathrm{sc}\left(\lambda x \mid 1 - \frac{1}{4}\dot{u}^2(0) \right) \right\}, \tag{4.2}$$

where $\dot{u}(0)$, the derivative of u at 0, is given by the expression $\dot{u}(0) = 2(1-m)^{1/2}$, with m being the solution of the transcendental equation

$$\frac{\sinh\left(\frac{\lambda}{2}\right)}{(1-m)^{1/2}} = \mathrm{sc}(\lambda \mid m), \tag{4.3}$$

where $\mathrm{sc}(\lambda \mid m)$ is the Jacobi elliptic function[3] (see, for example, [1, 14]). From (4.2), it was noted in [18] that a pole of $u(t)$ occurs at a pole of $\mathrm{sc}(\lambda x \mid 1 - \frac{1}{4}\dot{u}^2(0))$. It was also noted in [18] that the pole occurs at

$$x \approx \frac{1}{2\lambda} \ln\left(\frac{16}{1-m} \right). \tag{4.4}$$

[3] The Jacobi elliptic function $\mathrm{sc}(\lambda \mid m)$ is defined by $\mathrm{sc}(\lambda \mid m) = \frac{\sin\phi}{\cos\phi}$, where ϕ, λ, and m are related by the integral

$$\lambda = \int_0^{\phi} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}}.$$

It also has an equivalent definition given in terms of a lattice.

Here we will show how to apply the decomposition method to solve this problem. We will present the solution for $0 < \lambda \leq 1$. It is for these values of λ's that the method converges. For $\lambda > 1$, it follows from (4.4) that the pole of the exact solution $u(t)$, (i.e., the pole of $\mathrm{sc}(\lambda x \mid 1 - \frac{1}{4}\dot{u}^2(0)))$ occurs within the interval (0, 1). Thus for $\lambda > 1$, Conditions 1 and 2 for the convergence of the decomposition method stated in Section 2 will be violated. Indeed, the nonlinearity $\sinh(\lambda u)$ will not be analytic.

We first write the integral equation to this boundary value problem following Eq. (3.3),

$$u(x) = (1-x)\int_0^x -s\lambda \sinh(\lambda u)\, ds + x \int_x^1 -(1-s)\lambda \sinh(\lambda u)\, ds + x. \tag{4.5}$$

Let

$$u_0 = x.$$

We expand $\sinh(\lambda u)$ around u_0,

$$\begin{aligned}\sinh(\lambda u) &= \sinh(\lambda u_0) + \lambda \cosh(\lambda u_0)(u-u_0) + \frac{\lambda^2 \sinh(\lambda u_0)}{2!}(u-u_0)^2 \\ &\quad + \frac{\lambda^3 \cosh(\lambda u_0)}{3!}(u-u_0)^3 + \frac{\lambda^4 \sinh(\lambda u_0)}{4!}(u-u_0)^4 + \cdots,\end{aligned} \tag{4.6}$$

and also in terms of the Adomian polynomials as

$$\sinh(\lambda u) = A_0 + A_1 + A_2 + A_3 + A_5 + \cdots.$$

Observing that $u - u_0 = u_1 + u_2 + u_3 + \cdots$, we obtain the Adomian polynomials with the first six listed below:

$$\begin{aligned}
A_0 &= \sinh(\lambda u_0),\\
A_1 &= \lambda u_1 \cosh(\lambda u_0),\\
A_2 &= \lambda u_2 \cosh(\lambda u_0) + \frac{1}{2}\lambda^2 u_1^2 \sinh(\lambda u_0),\\
A_3 &= \lambda u_3 \cosh(\lambda u_0) + \lambda^2 u_1 u_2 \sinh(\lambda u_0) + \frac{1}{3!}\lambda^3 u_1^3 \cosh(\lambda u_0),\\
A_4 &= \lambda u_4 \cosh(\lambda u_0) + \lambda^2 u_1 u_3 \sinh(\lambda u_0) + \frac{1}{2!}\lambda^2 u_2^2 \sinh(\lambda u_0) + \frac{1}{2}\lambda^3 u_1^2 u_2 \cosh(\lambda u_0)\\
&\quad + \frac{1}{4!}\lambda^4 u_1^4 \sinh(\lambda u_0),\\
A_5 &= \lambda u_5 \cosh(\lambda u_0) + \lambda^2 (u_1 u_4 + u_2 u_4) \sinh(\lambda u_0) + \frac{1}{2}\lambda^3 \left(u_1^2 u_3 + u_1 u_2^2\right) \cosh(\lambda u_0)\\
&\quad + \frac{1}{5!}\lambda^5 u_1^5 \cosh(\lambda u_0),\\
&\ldots
\end{aligned}$$

Continuing this method, we can find $A_6,\ A_7$, etc. These expressions, along with Eq. (4.5), yield the series solution of Troesch's problem,

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \tag{4.7}$$

where $u_k(x)$ are given by the iteration scheme

$$u_0 = x,$$
$$u_{k+1}(x) = (1-x)\int_0^x -s\lambda A_k\,ds + x\int_x^1 -(1-s)\lambda A_k\,ds,$$

for $k = 1, 2, 3, \ldots$.

We used the computer algebra system, Maple V, to obtain the first six iterations in u. We list the first three terms:

$$u_0 = x,$$
$$u_1 = \frac{\sinh(\lambda x) - x\sinh(\lambda)}{\lambda},$$
$$u_2 = -\frac{1}{4\lambda^2}[-\lambda\cosh(\lambda x)\sinh(\lambda x) + 4\lambda x\sinh(\lambda)\cosh(\lambda x) - 8\sinh(\lambda)\sinh(\lambda x)$$
$$- 3\lambda x\sinh(\lambda)\cosh(\lambda) + 8x\cosh^2(\lambda) - 8x],$$
$$\vdots$$

The approximation is carried out for Troesch's problem with $\lambda = 0.5$ and $\lambda = 1$ at $x = 0.1, 0.2, \ldots$, and 1.0. Tables I and II exhibit the results of the approximation using only six terms in Eq. (4.7) for $\lambda = 0.5$ and 1. These tables also give the value of the exact solutions as given in the closed form (4.2). In (4.2), for a given λ, we use (4.3) to determine m and the expression $\dot{u}(0) = 2(1-m)^{1/2}$ to determine $\dot{y}(0)$. The last column of the table lists the error.

The errors in Table II are less than 1.3%. The decomposition method is immediate to apply and yields a reasonable approximation to the solution when $0 < \lambda \leq 1$ with only

TABLE I
Decomposition Method Approximation for $u'' = \lambda \sinh \lambda u$ ($\lambda = 0.5$), $u(0) = 0$, $u(1) = 1$

x	Exact solution	Numerical solution	Error
0.1	0.0951769020	0.0959383534	0.0007614514
0.2	0.1906338691	0.1921180592	0.0014841901
0.3	0.2866534030	0.2887803297	0.0021269267
0.4	0.3835229288	0.3861687095	0.0026457807
0.5	0.4815373854	0.4845302901	0.0029929047
0.6	0.5810019749	0.5841169798	0.0031150049
0.7	0.6822351326	0.6851868451	0.0029517125
0.8	0.7855717867	0.7880055691	0.0024337824
0.9	0.8913669875	0.8928480234	0.0014810369
1.0	0.9999999999	0.9999999988	0.0000000011

TABLE II
Decomposition Method Approximation for $u'' = \lambda \sinh \lambda u$ ($\lambda = 1$), $u(0) = 0$, $u(1) = 1$

x	Exact solution	Numerical solution	Error
0.1	0.0817969966	0.084248760	0.0024517634
0.2	0.1645308709	0.169430700	0.0048998291
0.3	0.2491673608	0.256414500	0.0072471392
0.4	0.3367322092	0.346085720	0.0093535108
0.5	0.4283471610	0.439401985	0.0110548241
0.6	0.5252740296	0.537365700	0.0120916704
0.7	0.6289711434	0.641083800	0.0121126566
0.8	0.7411683782	0.751788000	0.0106196218
0.9	0.8639700206	0.870908700	0.0069386794
1.0	1.0000000020	0.999998200	0.0000018020

six terms of the series, which are easy to compute using a computer algebra system (for example, Maple V). For $\lambda > 1$, we already noted that the decomposition method does not yield a good approximation because the exact solution $u(x)$ has a pole within the interval (0, 1).

Also, the graphs of the exact solution (solid curve) and the solution obtained by the decomposition method (dotted curve) are presented in Figs. 1 and 2.

EXAMPLE 2 (Bratu's problem). We consider the boundary value problem

$$-u'' = \lambda e^u \tag{4.8}$$

with the boundary conditions $u(0) = 0,\ u(1) = 0$.

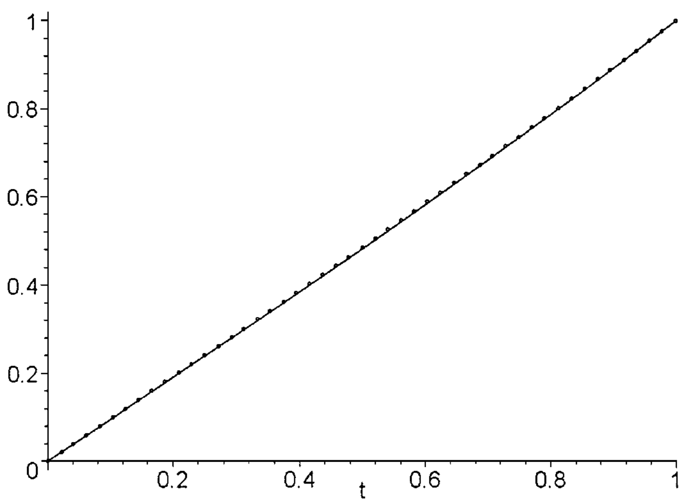

FIG. 1. Troesch's problem: decomposition method versus analytic solution ($\lambda = 0.5$).

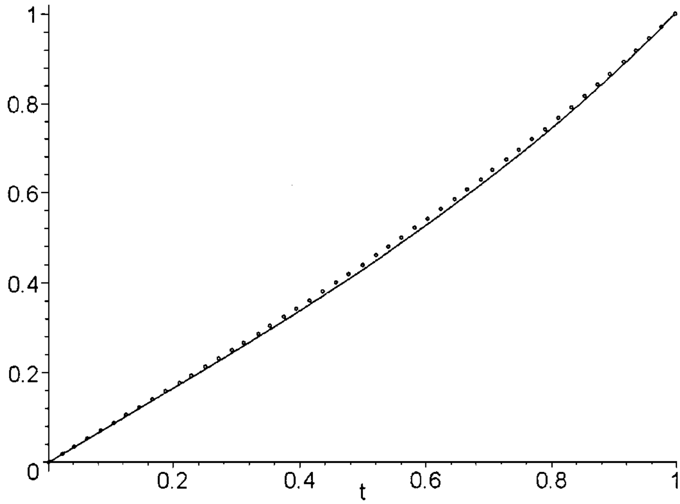

FIG. 2. Troesch's problem: decomposition method versus analytic solution ($\lambda = 1$).

This is referred to in the literature [8, 15] as Bratu's problem. In higher dimension, Eq. (4.8) models a combustion problem in a slab. It was noted in [9] that the function

$$u(x) = -2\log\left[\cosh\left(\frac{0.5(x-0.5)\theta}{0.25\cosh\theta}\right)\right] \tag{4.9}$$

is a solution to (4.8), provided θ is the solution of $\theta = \sqrt{2\lambda}\,\cosh(\theta/4)$. This equation has two, one, or no solution when $\lambda < \lambda_c$, $\lambda = \lambda_c$, $\lambda > \lambda_c$, where the critical value λ_c satisfies the equation $1 = \frac{1}{4}\sqrt{2\lambda}\,\sinh(\frac{\theta}{4})$. Numerical solutions of this problem were obtained by a shooting method (see [9, 17]).

Problem (4.8) can be represented in an integral form as in Eq. (3.3),

$$u(x) = \lambda\int_0^1 g(x,s)e^u\,ds,$$

where $g(x,s)$ is the Green's function given in (3.2). According to the analysis in Section 3, the solution $u(x)$ is represented by the series as in (3.4) with $u_0 = 0$.

The nonlinearity e^u may be expanded using Eq. (3.5),

$$e^u = \sum_{i=0}^{\infty} A_i = A_0 + A_1 + A_2 + \cdots.$$

From the Taylor series of e^u at u_0 (which is 0 in this example)

$$e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \cdots,$$

we can find

$$\begin{aligned} A_0 &= 1,\\ A_1 &= u_1 e^{u_0},\\ A_2 &= u_2 + \frac{1}{2!}u_1^2, \end{aligned}$$

$$A_3 = u_3 + u_1 u_2 + \frac{1}{3!} u_1^3,$$
$$A_4 = u_4 + u_1 u_3 + \frac{1}{2!} u_2^2 + \frac{1}{2} u_1^2 u_2 + \frac{1}{4!} u_1^4,$$
$$\vdots$$

Substitution of A_i $(i = 0, 1, 2, \ldots)$ into Eq. (3.7) yields the values of $u_1, u_2, \ldots, u_{n+1}, \ldots$. Then the solution $u(x) = u_0 + u_1 + u_2 + \cdots$ can be determined by Eq. (3.6). Using the computer algebra system Maple V, we can obtain the first several terms. We list only the first three terms:

$$u_1 = -\frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda x,$$
$$u_2 = \frac{1}{24}\lambda^2 x^4 - \frac{1}{12}\lambda^2 x^3 + \frac{1}{24}\lambda^2 x,$$
$$u_3 = -\frac{1}{180}\lambda^3 x^6 + \frac{1}{60}\lambda^3 x^5 - \frac{1}{96}\lambda^3 x^4 - \frac{1}{144}\lambda^3 x^3 + \frac{1}{160}\lambda^3 x,$$
$$\vdots$$

We shall consider, for example, the case $\lambda = 1$. In this case, problem (4.8) has two locally unique solutions u_1 and u_2 with $u_1'(0) \approx 0.549$ and $u_2'(0) \approx 10.909$ (see [9]). The solution of the decomposition method given by $u_0 + u_1 + u_2 + u_3 + \cdots$ converges to the solution $u_1'(0) \approx 0.549$, and not to the solution $u_2'(0) \approx 10.909$.

In Table III we compare the exact solution derived from Eq. (4.9) with the numerical solution obtained by the decomposition method using only four terms at $x = 0.1, 0.2, \ldots, 1.0$ for $\lambda = 1$.

We observe that the error is less than 0.31%.

The accuracy of the approximation is also reflected in Fig. 3, in which the solid curve represents the analytic solution, while the dotted curve is the approximation solution. We can observe the almost perfect match of these two solutions.

TABLE III

Decomposition Method Approximation for $u'' = \lambda e^u$ ($\lambda = 1$), $u(0) = u(1) = 0$

x	Exact solution	Numerical solution	Error
0.1	0.0498467900	0.0471616875	0.0026851025
0.2	0.0891899350	0.0871680000	0.0020219350
0.3	0.1176090956	0.1177614375	0.0001523419
0.4	0.1347902526	0.1369920000	0.0022017474
0.5	0.1405392142	0.1435546875	0.0030154733
0.6	0.1347902526	0.1369920000	0.0022017474
0.7	0.1176090956	0.1177614375	0.0001523419
0.8	0.0891899350	0.0871680000	0.0020219350
0.9	0.0498467900	0.0471616875	0.0026851025

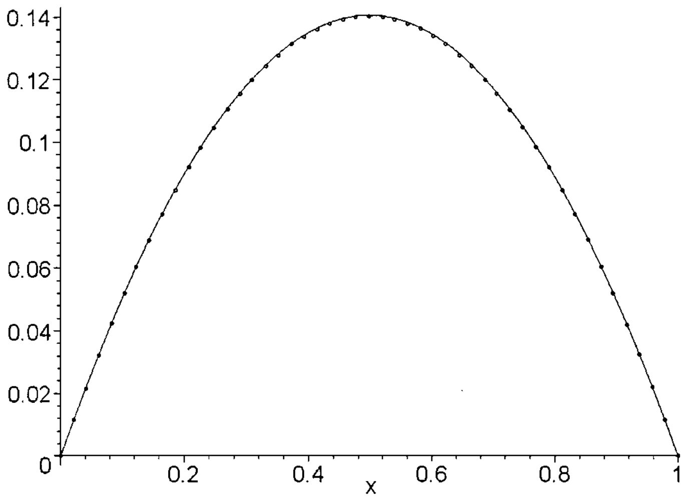

FIG. 3. Bratu's problem: decomposition method versus analytic solution ($\lambda = 1$).

For $\lambda < 1$, the result with only four terms is even better than 0.3%. As λ approaches the critical value λ_c, the error becomes larger, and the convergence becomes slower as shown in the case of $\lambda = 2$ below. This is due to the fact that the decomposition method yields solutions that converge to one of the solutions of problem (4.8). As we noted earlier, in the case of $\lambda = 1$, the solution converges to the solution with the initial condition $u_1'(0) \approx 0.549$. In summary, the decomposition method with only four terms seems to give a very reasonable approximation, and the terms can be easily computed using a computer algebra system, e.g., Maple V.

Table IV and Fig. 4 show the analytic and the the decomposition method solutions obtained by the decomposition method for $\lambda = 2$. We observe that the error is about 1%, and Fig. 4 shows that the discrepancy between the analytic and the approximate solution starts to aggravate.

In this paper we presented the decomposition method as an alternate method to the shooting method to solve two important boundary value problems. In both problems, the

TABLE IV
Decomposition Method Approximation for $u'' = \lambda e^u$ ($\lambda = 2$), $u(0) = u(1) = 0$

x	Exact solution	Numerical solution	Error
0.1	0.0991935000	0.1144107440	0.0152172440
0.2	0.1917440000	0.2064191156	0.0146751156
0.3	0.2679915000	0.2738793116	0.0058878116
0.4	0.3183360000	0.3150893646	0.0032466354
0.5	0.3359375000	0.3289524214	0.0069850786
0.6	0.3183360000	0.3150893646	0.0032466354
0.7	0.2679915000	0.2738793116	0.0058878116
0.8	0.1917440000	0.2064191156	0.0146751156
0.9	0.0991935000	0.1144107440	0.0152172440

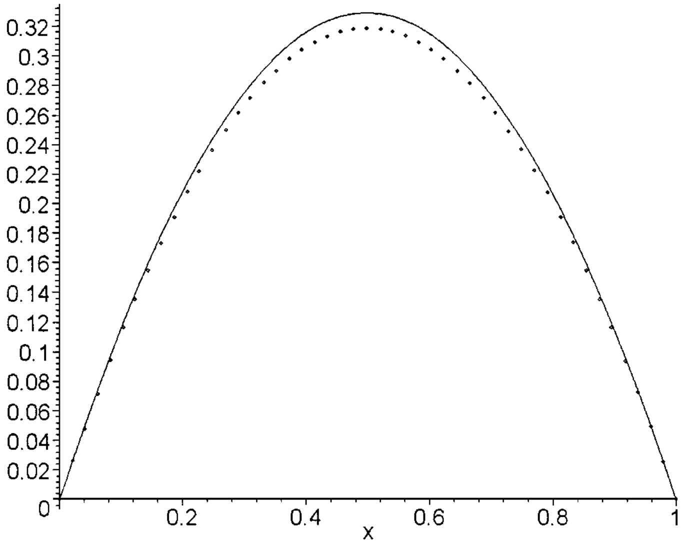

FIG. 4. Bratu's problem: decomposition method versus analytic solution ($\lambda = 2$).

method yields accurate computable solutions with good approximation using only a few terms, provided that the parameter λ satisfies $0 < \lambda \leq 1$.

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